**Indian Statistical Institute, Bangalore Centre**
**B.Math. (III Year) : 2012-2013**
**Semester I : Semestral Examination**
**Probability III (Stochastic Processes)**

26.11.2012 Time: 3 hours. Maximum Marks : 100

*Note:* The paper carries 105 marks. Any score above 100 will be taken as 100. Notation and terminology are understood to be as used in class. State clearly the results you are using in your answers.

1. ( $10 + 5 = 15$ marks) Let $\xi_1, \xi_2, \cdots$ denote the outcomes of independent rolls of a fair die. Let $X_n = \max\{\xi_k : k = 1, 2, \cdots, n\}, n \geq 1$.

   (i) Show that $\{X_n : n \geq 1\}$ is a Markov chain, and find its transition probability matrix.

   (ii) Find the recurrent and transient states.

2. ( $10 + 5 = 15$ marks) (i) For the Ehrenfest urn model with a total of $2d$ balls, show that the binomial distribution with parameters $2d$ and $\frac{1}{2}$ is the unique stationary probability distribution.

   (ii) Show that the Markov chain in (i) is time-reversible.

3. ( $12 + 18 = 30$ marks) (i) Let $\{Z_n : n \geq 0\}$ be a Markov chain on a countable state space $S$ with transition probability matrix $Q = ((Q_{ij}))$. Suppose $Z_0 = x \in S$. Let $H_x = \min\{n \geq 1 : Z_n \neq x\}$. Show that $H_x - 1$ has a geometric distribution, and find $E(H_x \mid Z_0 = x)$.

   (ii) Suppose that a production process changes states in accordance with a Markov chain $\{X_n\}$ on $S = \{1, 2, 3, 4\}$ with transition probability matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

   Let $A = \{1, 2\}, A^c = \{3, 4\}$. If the production process is in a state belonging to $A$ then it is said to be "up", otherwise, it is said to be "down". Assume that the production process has been operating for a long time. Find the average time the process remains "up" once it is in an "up" state, and the average time the process remains "down" once it is in a "down" state.

4. ( $10 + 9 + 6 = 25$ marks) (i) Let $\{N(t) : t \geq 0\}$ be a time-homogeneous Poisson process with rate $\lambda > 0$. For $n = 1, 2, \cdots$ let $W_n$ be the waiting time until the $n$-th event. Show that $P(W_n < \infty) = 1$ for any $n$.

   (ii) With $W_n$ as in (i), find the distribution function and the probability density function of $W_n, n \geq 1$.

   (iii) Suppose shocks in a device happen according to a time homogeneous Poisson process with rate $\lambda > 0$. Assume that the device fails when a cumulative effect of $k$ shocks occurs. Let $T$ denote the lifetime of the device. Find the probability density function of $T$.

5. ( $12 + 8 = 20$ marks) (i) Let $\{X(t) : t \geq 0\}$ be a (time homogeneous) continuous-time Markov chain with state space $\mathbb{Z}$; assume that the sample paths of $X$ are right continuous. Suppose $X(0) = i$; denote $\tau_i = \inf\{t > 0 : X(t) \neq i\}$. Show that $\tau_i$ has an exponential distribution.

   (ii) Let $X$ be as in (i); let $(i, j, t) \mapsto P_{ij}(t), i, j \in \mathbb{Z}$ denote the transition probability function of $X$. Show that

$$P_{ij}(t + s) = \sum_k P_{ik}(t) P_{kj}(s), \quad i, j \in \mathbb{Z}, s, t \geq 0.$$